# SOME TWO-DIMENSIONAL FLOWS AT FINITE MAGNETIC REYNOLDS NUMBERS

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Flows at finite magnetic Reynolds numbers are characterized by a strong effect of the induced magnetic fields on the stream. In the present paper we determine the current distribution and estimate the influence of the Lorentz force component perpendicular to the stream in a two-dimensional channel with electrodes. We also estimate the influence of nonuniformities of the velocity in the stream path of an incompressible fluid when the characteristic magnetic Reynolds numbers are not small.

**1. Some properties of induced magnetic fields in a two-dimensional channel.** The problem of determining the induced magnetic field in magnetohydrodynamic flow in a two-dimensional channel has been solved by several authors (e. g., see [1–4]). In [1–3] the magnetic field is determined from the given current distribution, even though the current in fact depends on the magnetic field. We propose to solve this problem for a two-dimensional channel of finite dimensions by using a somewhat different mathematical approach. We shall find the current distribution and magnetic field assuming that the flow velocity $u_0$ (along the x axis) is constant.

Let $H_1$ and $H_3$ be the x- and z-components, respectively, of the induced magnetic field; $H_0$ is the external field, which is directed along the z axis; L is the length of the channel; b is the dimension of the channel along the external magnetic field; $l$ is the distance between electrodes.

The induced magnetic field is defined by the following system (which does not take account of the dependence on y):

$$H_{3x} - H_{1z} = R(H_3 + H_0), \quad H_{1x} + H_{3z} = 0,$$

$$H_{3x} = \partial H_3/\partial x \quad \text{etc.}$$

$$R = 4\pi\sigma u_0$$

For simplicity we assume throughout this paper that the electrical circuit has been shorted out, that the external magnetic field is given, and that it is produced without the aid of ferromagnetic materials.

Setting

$$H_1 = -A_z, \ H_3 = A_x$$

we obtain

$$\triangle A = R(A_x + H_0), \quad \triangle = \partial^2/\partial x^2 + \partial^2/\partial z^2 . \quad (1.1)$$

The right side of Eq. (1.1) is proportional to the current, so that its solution is

$$A = \lambda \int_0^L \int_0^b \ln \sqrt{(x-x')^2 + (z-z')^2} \times$$

$$\times (A_{x'} + H_0)\, dx'\, dz',$$

$$\lambda = \frac{R}{2\pi} .$$

The field component $H_3$ is defined as the solution of the Fredholm integral equation

$$H_3 = \lambda \int_0^L \int_0^b \frac{(x-x')(H_0 + H_3)}{(x-x')^2 + (z-z')^2}\, dx'\, dz' . \quad (1.2)$$

under the condition

$$\lambda \left| \int_0^L \int_0^b K\, dx'\, dz' \right| < 1 .$$

Here K is the kernel of integral equation (1.2), which can be solved by the method of successive approximations.

The first approximation yields

$$H_3 = \lambda \int_0^L \int_0^b \frac{(x-x')H_0}{(x-x')^2 + (z-z')^2}\, dx'\, dz' .$$

When $b/L \ll 1$ the component $H_3$ in this approximation depends weakly on z and is odd in x = L/2. For values of x not too close to the boundary values we have

$$H_3 = \lambda b H_0 \ln \frac{x}{L-x} .$$

The component $H_1$ in the first approximation is

$$H_1 = \lambda H_0 \int_0^L \int_0^b \frac{(z'-z)\, dx'\, dz'}{(x-x')^2 + (z-z')^2} .$$

The function $H_1(x, z)$ is even in x = L/2 and odd in z = b/2. For small b/L the component $H_1$ is linear in z,

$$H_1 = R\left(\tfrac{1}{2} b - z\right) H_0 ,$$

if the values of x are not close to the boundary values. At the channel boundaries we have

$$H_1 = \tfrac{1}{2} R \left(\tfrac{1}{2} b - z\right) H_0 \quad \text{for } x = 0, L .$$

In the other limiting case, i. e., when $b \gg L$, we can readily verify by direct substitution that for values of z not close to the boundary values Eq. (1.2) has the solution

$$H_3 = H_0 \left( \frac{2 \exp Rx}{\exp RL + 1} - 1 \right). \quad (1.3)$$

This agrees entirely with the solution obtained in [5].

As $\lambda \to \infty$ the integrand in (1.2) must tend to zero (i. e., $H_3 \to -H_0$), since $H_3$ must remain a finite quantity. This conclusion is valid, however, without allowance for the magnetic field set up by the current in the external electrical circuit.

**2. Appearance of non-one-dimensional flow in a narrow channel.** The above approximation is valid only if the flow velocity $u_0$ is constant. However, the presence along the z axis of a force

$$F = -j_0 H_1$$

(where $j_0$ is the component of the current density along the y axis) leads to the appearance of a velocity component w (along the z axis).

The appearance of non-one-dimensional flow was considered by Pitkin [3], who estimated the variation of the longitudinal component of the velocity. He assumed, however, that such flow could be described by means of the Bernoulli equation. Making use of the above approximate expressions for the fields when $Rb \lesssim 1$ in the case of a channel such that $b/L \ll 1$, we can show (see Appendix), that a strong nonuniformity arises only at the channel boundaries for x = 0, L; at points far away from the boundaries the force F is compensated by the pressure gradient, and the ratio $w/u_0$ is small if $L \gg L_M = \rho u_0/\sigma H_0^2$, i. e., if the channel is long enough to enable the magnetic field to act on the stream.

We note that the ratio of the longitudinal component of the Lorentz force to the transverse component is $H_1/H_2 \sim Rb$, so that $Rb \ll 1$ is a sufficient condition for the effect of the induced fields on the stream to be small.

**3. Flow in a wide channel with a one-dimensional velocity profile.** In the presence of a magnetic field, small gradients of the density or velocity of the medium along the current path can result in considerable distortion of the streamlines. This is because the dimensionless parameters characterizing the effect of the medium on the current flow through it (e. g., the product of the electronic cyclotron frequency and the average time between collisions for an electron, or the magnetic Reynolds number) can generally exceed unity (e. g., see [6–8]). Steenbeck [8] investigates the reduction of the current flowing between electrodes in the presence of chaotic turbulence of the medium as compared to the current which flows when the liquid is at rest; he carries out his analysis for a certain specified distribution of the velocity correlations in an incompressible fluid. We propose to investigate the effect of velocity nonuniformities on the current during flow of a fluid in a wide ($b \gg L$) channel of constant cross section. Let the velocity component u(y) be given and let it be independent of the magnetic field. This is the case, for example, when metal strips move at different velocities.

We represent the velocity component u as $u = u_0 + u_1$, where $u_0$ is the average value of u. Motion with the velocity $u_0$ produces the magnetic field $H_3$ defined by relation (1.3). The presence of a fluctuating component $u_1$ gives rise to an electric field with the potential $\varphi(x, y)$ and to a magnetic field $h_3(x, y)$. The problem of velocity nonuniformities along the stream path in the case of small Reynolds numbers is solved numerically in [9].

From the equations of magnetohydrodynamics for flow with the velocity u(y) we have

$$h_{3y} = 4\pi\sigma\varphi_x,$$

$$H_{3x} + h_{3x} + 4\pi\sigma[\varphi_y - u(H_0 + H_3 + h_3)] = 0. \qquad (3.1)$$

The functions $\varphi$ and $h_3$ must satisfy the following conditions: $\varphi = 0$ for $y = 0, l$; $\varphi_x = 0$ for $x = 0, L$; $h_{3y} = 0$ for $y = 0, l$.

The case of small magnetic Reynolds numbers is simplest. Here system (3.1) becomes

$$h_{3y} = 4\pi\sigma\varphi_x, \qquad H_{3x} + h_{3x} + 4\pi\sigma(\varphi_y - uH_0) = 0.$$

Eliminating the magnetic field, we obtain the equation for the potential:

$$\Delta\varphi = u_y H_0.$$

The solution of this equation does not depend on x and is of the form

$$\varphi = \int_0^y uH_0 dy - \frac{y}{l}\int_0^l uH_0 dy.$$

This implies that

$$I_1 = 0, \qquad I_2 = -\sigma H_0 \langle u \rangle, \qquad \langle u \rangle = \frac{1}{l}\int_0^l u(y)\, dy.$$

Here $I_1$ and $I_2$ are the components of the current density.

For arbitrary magnetic Reynolds numbers RL we consider the influence of the additional component $u_1$ as a perturbation. This is valid provided the relative velocity fluctuation and the ratio of the fluctuation-induced magnetic field to the sum magnetic field are small. In linear approximation system (3.1) becomes

$$h_{3y} = 4\pi\sigma\varphi_x,$$

$$h_{3x} + 4\pi\sigma[\varphi_y - u_0 h_3 - u_1(H_0 + H_3)] = 0.$$

The function $u_1$ can be expanded in some complete system of functions, e. g., in the system $\cos n\pi y/l$. The potential distribution in this case is of the form

$$\varphi = \sum_n \varphi_n(x) \sin\frac{n\pi y}{l}.$$

The current corrections due to the n-th harmonic of the perturbation $u_{1n}$ are

$$\frac{j_1}{j_0} = \frac{Ru_0}{\alpha^2}\frac{du_{1n}}{dy}\{1 + A_1 \exp[(k_1 - R)x] + A_2 \exp[(k_2 - R)x]\}, \qquad \frac{j_2}{j_0} = -\frac{u_{1n}R^2}{u_0\alpha^2} \times$$

$$\times\left\{1 + A_1\frac{k_1}{R}\exp[(k_1 - R)x] + A_2\frac{k_2}{R}\exp[(k_2 - R)x]\right\}, \quad j_0 = -\sigma u_0(H_3 + H_0),$$

$$\alpha = \frac{n\pi}{l}, \qquad k_{1,2} = {}^1\!/_2 R \pm \sqrt{{}^1\!/_4 R^2 + \alpha^2},$$

$$A_1 = \frac{\exp RL - \exp k_2 L}{\exp k_2 L - \exp k_1 L}.$$

Here $j_0$ is the value of the current in the absence of fluctuations, and $A_2$ can be found from $A_1$ by interchanging subscripts.

Let us consider the case of large-scale fluctuation and large RL. Let the fluctuation along the y axis be sufficiently large, so that $R^2/\alpha^2 \gg 1$, but at the same time let $\exp(\alpha^2 L/R) \gg 1$ (it is clear here that $\exp RL \gg 1$). Then for values of x not too close to the boundary values we have

$$\frac{j_1}{j_0} = \frac{R}{\alpha^2 u_0}\frac{du_{1n}}{dy}, \qquad \frac{j_2}{j_0} = -\frac{R^2}{\alpha^2}\frac{u_{1n}}{u_0}.$$

Let us consider the other limiting case where the fluctuation along y is so large that

$$R/\alpha \gg 1, \quad \alpha^2 L/R \ll 1$$

(as above, $\exp RL \gg 1$).

We then obtain the following expressions for the current corrections for values of x not close to the limiting values:

$$\frac{j_1}{j_0} = \frac{1}{u_0}\frac{du_{1n}}{dy}(L-x),$$

$$\frac{j_2}{j_0} = \frac{u_{1n}}{u_0}[1 + R(x-L)].$$

As is evident from the limiting cases considered above, the current corrections can be substantial even with small relative velocity fluctuations if the magnetic Reynolds numbers determined both by channel length and by the fluctuation scale are large.

However, the ratio of the energy dissipation due to the fluctuation component $u_1$ to the sum dissipation,

$$(\langle j_1^2\rangle + \langle j_2^2\rangle)/j_0^2,$$

is on the order of the square of the relative velocity fluctuation without a factor containing the magnetic Reynolds number.

**4. Propagation of two-dimensional velocity fluctuations in a wide channel.** Let us investigate the propagation of two-dimensional fluctuations by the method of [10]. We assume that the perturbations of the velocity and pressure at the channel entrance are given, and consider their propagation downstream. Eliminating the pressure from the equations of continuity and motion

$$\operatorname{div}\mathbf{v} = 0, \qquad \rho(\mathbf{v}\nabla)\mathbf{v} = -\nabla p + \mathbf{j}\times\mathbf{H}$$

we obtain

$$u_{1x} + v_y = 0, \qquad u_{1xy} - v_{xx} = 0. \tag{4.1}$$

Here u, v are the components of the velocity perturbation.

As above, we expand the quantities which vanish at the electrodes in the system of functions $\sin n\pi y/l$. Let us consider the propagation of the large-scale fluctuation for $n = 1$. We seek a solution of the form

$$v = v(x)\sin\beta y,$$

$$u_1 = u_1(x)\cos\beta y, \qquad \beta = \pi/l.$$

From system (4.1) we obtain

$$v = (C_1 e^{\beta x} + C_2 e^{-\beta x})\sin\beta y,$$

$$u_1 = (C_2 e^{-\beta x} - C_1 e^{\beta x} + C_3)\cos\beta y.$$

The coefficients $C_i$ are defined by the given values of the fluctuations at the channel input:

$$C_1 + C_2 = v(0), \quad C_2 + C_3 - C_1 = u_1(0),$$

$$C_1 - C_2 = p_1(0)/\rho u_0.$$

The last equation follows from the linearized equation of motion

$$\rho u_0 u_x = -p_{1y} - j_1(H_0 + H_3)$$

if we take account of the condition $j_1 = 0$ for $x = 0$.

We can determine the distribution of the magnetic fields and potential from the system

$$h_{3y} = 4\pi\sigma[\varphi_x + v(H_0 + H_3)],$$

$$h_{3x} + 4\pi\sigma[\varphi_y - u_1(H_0 + H_3) - u_0 h_3] = 0. \tag{4.2}$$

The boundary conditions for the functions $\varphi$ and $h_3$ are similar to the conditions considered previously. From the solution of system (4.2) we have

$$h_3 = -\frac{4\pi\sigma}{\beta}\Big[(H_0 + H_3)\Big(\frac{R}{\beta}C_3 + C_1 e^{\beta x} + C_2 e^{-\beta x}\Big) + $$

$$+ C_4 e^{\gamma_1 x} + C_5 e^{\gamma_2 x}\Big]\cos\beta y,$$

$$\gamma_{1,2} = {}^1\!/_2 R \pm \sqrt{{}^1\!/_4 R^2 + \beta^2}. \tag{4.3}$$

The values of the constants $C_4$ and $C_5$ can be determined from the condition

$$h_{3y} = 0 \text{ for } x = 0,\ L.$$

The function $h_3$ can generally be determined from (4.3) to within the term $C \exp Rx$, where C is a constant. This term is omitted in expression (4.3). Recalling that the potential difference across the electrodes is independent of C (by virtue of (4.2)), we assume that the current due to the variation of $h_3$ is equal to zero, i. e., that $C = 0$.

The current perturbation components can be obtained by differentiating $h_3$:

$$j_1 = \sigma\Big[(H_0 + H_3)\Big(C_1 e^{\beta x} + C_2 e^{-\beta x} + \frac{R}{\beta}C_3\Big) + $$

$$+ C_4 e^{\gamma_1 x} + C_5 e^{\gamma_2 x}\Big]\sin\beta y,$$

$$j_2 = \sigma\Big\{\Big[C_1\Big(1 + \frac{R}{\beta}\Big)e^{\beta x} + C_2\Big(\frac{R}{\beta} - 1\Big)e^{-\beta x} + $$

$$+ \frac{R^2}{\beta^2}C_3\Big](H_0 + H_3) + C_4\frac{\gamma_1}{\beta}e^{\gamma_1 x} + C_5\frac{\gamma_2}{\beta}e^{\gamma_2 x}\Big\},$$

$$C_4 = \frac{f(L) - f(0)\exp\gamma_2 L}{\exp\gamma_2 L - \exp\gamma_1 L},$$

$$f(x) = (H_0 + H_3)\Big(\frac{RC_3}{\beta} + C_1 e^{\beta x} + C_2 e^{-\beta x}\Big). \tag{4.4}$$

The constant $C_5$ can be determined from $C_4$ by interchanging subscripts.

The above example of a one-dimensional velocity profile can be obtained from this solution by setting $C_1 = C_2 = 0$.

Expressions for the current perturbations (4.4) imply that a similar picture of strong current distortion results from velocity perturbations in the direction of the current. For example, let $C_2 = C_3 = 0$. In the limiting case $\exp RL \gg 1$, $R/\beta \gg 1$, $\beta^2 L/R \ll 1$ the expressions for current corrections (4.4) become (for values of x not close to the boundary values)

$$\frac{j_1}{j_0} = \frac{C_1}{u_0}(e^{\beta L} - e^{\beta x}), \qquad j_2 = \frac{R}{\beta} j_1 .$$

The above examples, therefore, show that the flow of an incompressible fluid in a channel at large magnetic Reynolds numbers, the propagation of perturbations in the channel, and the effect of velocity inhomogeneities on the current are determined to a large extent by the dimensions of the channel and by the conditions at its boundaries (by the presence or absence of ferromagnetic materials). In a wide channel (a channel with a one-dimensional magnetic field) small fluctuations of the velocity along the current path result in strong distortion of the current. In a narrow channel the induced magnetic fields result in nonuniform flow at the channel entrance and outlet with an increasing minimal magnetic Reynolds number.

5. **Appendix.** Let us consider the effect of induced magnetic fields as a perturbation and write out a linearized system of equations for the velocity perturbations $u_1$, $w$ and the pressure perturbation $p_1$:

$$\rho u_0 u_{1x} = -p_{1x} - 2\sigma u_0 H_0 H_3 - \sigma u_1 H_0^2,$$

$$\rho u_0 w_x = -p_{1z} + \sigma u_0 H_1 H_0, \quad u_{1x} + w_z = 0 .$$

Let us assume that $H_3$ depends only on x and is odd in L/2, and that $H_1$ depends only on z. Eliminating $u_1$ and $p_1$, we obtain an equation for w:

$$w_{xxx} + w_{zzx} + k_0 w_{zz} = 0, \quad k_0 = \sigma H_0^2 / \rho u_0 = 1/L_M .$$

Since w must vanish at the channel wall, the solution of this equation can be written as

$$w = \sum_{s,\,n} C_s \exp k_s x \sin \tau z,$$

$$\tau = n\pi / b, \qquad k_s^3 = \tau^2 (k_0 + k_s) . \qquad (5.1)$$

Here $C_s$ are some constants. Summation over s is from unity to three. The functions $u_1$ and $p_1$ can be determined by integration over x:

$$u_1 = u_1(0, z) + \sum_{s,\,n} \frac{C_s \tau}{k_s}[1 - \exp(k_s x)] \cos \tau z$$

$$p_1 = p_1(0, z) - \sigma H_0^2 u_1(0, z) x - 2\sigma u_0 H_0 \int_0^x H_3 dx -$$

$$-\sigma H_0^2 \sum_{s,\,n} \frac{C_s \tau}{k_s}\left\{x + [1 - \exp(k_s x)]\left(\frac{1}{k_s} + \frac{1}{k_0}\right)\right\} \cos \tau z .$$

To determine the coefficients $C_s$ we make use of the conditions of continuity of the pressure and normal velocity component at the channel entrance and outlet. For example, let us specify the values of the perturbations $u_1$ and $p_1$ at the entrance and the pressure perturbation $p_1$ at the outlet (for simplicity we set them equal to zero, since these values are determined by conditions outside the channel). This yields the following system for the coefficients $C_s$:

$$\sum_s \frac{C_s}{k_s} = 0, \quad \sum_s C_s k_s \exp k_s L = k_0 u_0 \frac{H_{1n}}{H_0},$$

$$\sum_s C_s k_s (\exp k_s L - 1) = 0 .$$

Here $H_{1n}$ are the coefficients of the expansion of the function $H_1$ in the system of functions $\sin n\pi z/b$.

For $\tau \gg k_0$ the expression for w is

$$\frac{w}{u_0} = \sum_n \frac{k_0}{\tau} \frac{H_{1n}}{H_0}\left[e^{\tau(x-L)} - e^{-\tau x} + \frac{k_0}{\tau} e^{-k_0 x}\right] \sin \tau z .$$

The last term in the expression for w is important for values of x not close to the boundary values (x, L − x ≫ b). However, for $k_0 x \gg 1$ this term is also exponentially small. The maximum value of w is at the boundary:

$$\frac{w(0, z)}{u_0} = \sum_n \frac{k_0 R}{\tau^2}[1 + (-1)^n] \sin \tau z .$$

The condition $\tau \gg k_0$ generally need not be fulfilled for moderate values of n. In this case the characteristic equation (5.1) has one positive root $k_1$ and two complex conjugate roots $k_{2,3} = \varkappa_1 \pm i\varkappa_2$ with a negative real part (e.g., for $k \ll k_0$, $k = (k_0 \tau^2)^{1/3}$. Let the condition $\tau \gg k_0$ not be fulfilled for n = 1 (which means, of course, that $L \gg L_M$). We denote by $w_1$ the first term in the expansion of w in the system of functions $\sin n\pi z/b$:

$$w_1 = (S_1 e^{k_1 x} + S_2 e^{\varkappa_1 x} \sin \varkappa_2 x + S_3 e^{\varkappa_1 x} \cos \varkappa_2 z) \sin \tau z .$$

If we apply the same boundary conditions, the coefficient $S_1$ is proportional to $\exp(-k_1 L)$. Hence, even in the case $b \gg L_M$ for $L \gg L_M$ the component $H_1$ gives rise to non-one-dimensional flow only at the channel boundaries, all three exponentials in the expression for $w_1$ decay, and the perturbation $w_1$ diminishes far away from the channel boundaries.

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